Rotational Motion

1. Types of motion involving rotation -
(i) Rotation about a fixed axis - fan.
(ii) Rotation about an axis in translation < rolling, earth
(iii) Rotation about an axis in rotation - top
AKA precession.

2. It is not necessary that the axis of rotation should pass through the body.

3. All the particles of the body undergo same angular displacement in the same time interval, i.e., all particles of the body move with the same angular velocity (ω) and angular acceleration (α).

4. **Rigid Body** - a body whose constituent particles do not change their mutual distances under any circumstances. (the body is not deformed under any condition).
$\therefore$ no two particles in the same plane perpendicular to the axis of rotation have same velocity and acceleration vectors.

5. $\vec{\omega} = \frac{d\vec{\theta}}{dt}$ $\quad \vec{V} = \vec{\omega} \times \vec{r}$ $\quad \left(\omega = \frac{2\pi}{T} = 2\pi f\right)$

6. The angular velocity of a rigid body can be either +ve or -ve, depending on whether it is rotating in the direction of increasing θ (anticlockwise) or decreasing (θ) clockwise.

7. $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$ $\quad \vec{a_t} = \vec{\alpha} \times \vec{r}$ $\quad \vec{a_r} \, (a_c) = \vec{\omega} \times \vec{V}$

$\vec{a} = \vec{a_t} + \vec{a_r}$.

8. If $\alpha = 0 \Rightarrow \omega = \text{constant}$ and $\theta = \omega t$.

⇒ If α = constant -

(i) $\omega = \omega_0 + \alpha t$

(ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$.

(iii) $\omega^2 = \omega_0^2 + 2\alpha\theta$

(iv) $\theta_{n_{th}} = \omega_0 + \frac{\alpha}{2}(2n-1)$.

⇒ If α = not constant -

(i) $\omega = \frac{d\theta}{dt}$

(ii) $\alpha = \omega \frac{d\omega}{d\theta}$

9. Moment of Inertia - $\boxed{I = mr^2}$ ⇒ for point mass.

$\boxed{I = \int r^2 dm}$

- If $r\uparrow \Rightarrow I\uparrow$.

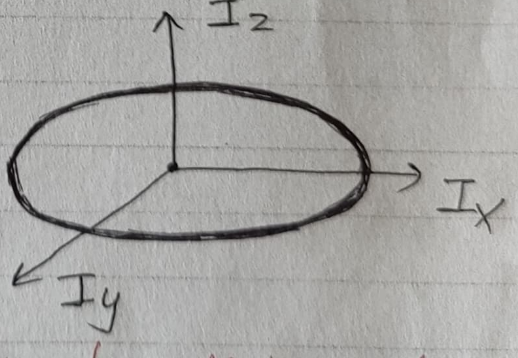
I depends on - (i) mass of the body
(ii) mass distribution of the body
(iii) Position of axis of rotation.

10. Radius of Gyration (K) - $\boxed{I = mk^2}$

11. Theorems of MoI -

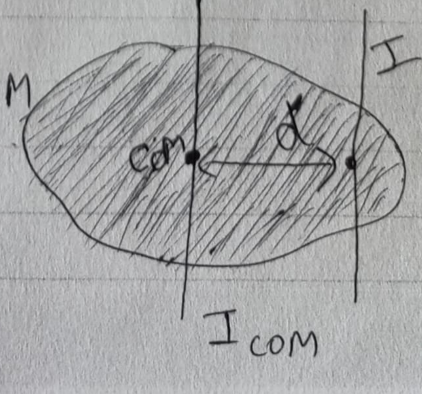
(i) Perpendicular Axis Theorem (applicable only to planar/2D/laminar bodies).

$\boxed{I_z = I_x + I_y}$



(ii) Parallel Axis Theorem (applicable to all body types).

$\boxed{I = I_{COM} + Md^2}$



12. **Torque** – rotational analogue of force
└ AKA moment of force.

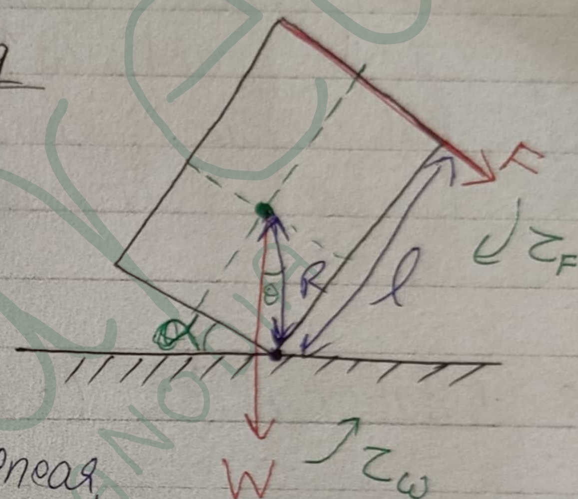
$\vec{\tau} = \vec{r} \times \vec{F}$.
$\vec{\tau} = I\vec{\alpha}$.

$\left\{ \begin{array}{l} \text{anticlockwise} = +ve \\ \text{clockwise} = -ve \end{array} \right\}$

13. **Rotational Equillibrium** $(\alpha = 0)$ ω = constant or 0.

$\Sigma \vec{\tau} = 0$.

14. **Condition for Rotation on Tilting**



$\tau_F > \tau_\omega$

$\Rightarrow Fl > WR\sin\theta$.

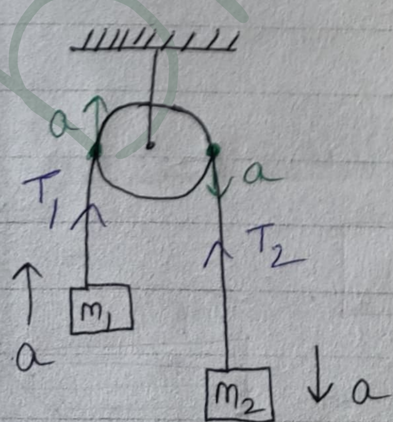
15. **Angular Momentum** = moment of linear momentum
$\vec{p} = m\vec{v}$.

$\boxed{\vec{L} = \vec{r} \times m\vec{v}}$ $\boxed{\vec{L_z} = I_z\vec{\omega}}$ (z = axis of rotation).

$\rightarrow \dfrac{d\vec{L}}{dt} = I\dfrac{d\vec{\omega}}{dt} = I\vec{\alpha} = \vec{\tau}$. $\left(\dfrac{d\vec{p}}{dt} = \vec{F}\right)$.

$\rightarrow$ angular impulse = $\vec{\tau}dt$.

16.



$T_1 \neq T_2$ The string does not slip on the pulley $\Rightarrow$ the pulley rotates ($\because$ net torque on the pulley $\neq 0$).

{ For ideal massless pulley, string slips. $T_1 = T_2$. Pulley does not rotate. $\Sigma\tau = 0$

17. Conservation of angular Momentum (COAM)

If $\Sigma \vec{\tau} = 0 \Rightarrow \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{constant}$.

$\Rightarrow L_i = L_f \Rightarrow \boxed{I_1 \omega_1 = I_2 \omega_2}$.

→ Examples of COAM - (i) Spinning Ice skater
(ii) student on rotating turntable.

* When student draws his arms towards himself $I \downarrow$ and $\omega \uparrow$. But KE also $\uparrow$.
But energy is conserved ∴ work done by student in bringing his arms inwards should also be accounted for.

18. Rotational KE (scalar) $\boxed{KE = \frac{1}{2} I \omega^2}$.

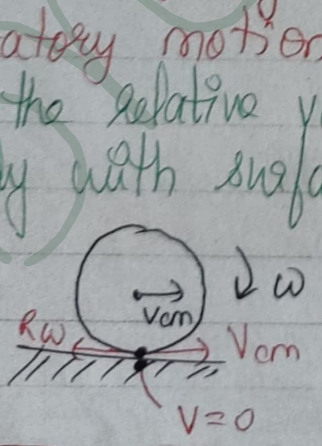
$$KE = \frac{1}{2} I \omega^2 = \frac{L^2}{2I} = \frac{L\omega}{2} = \frac{1}{2} M k^2 \omega^2 = \frac{1}{2} I \frac{4\pi^2}{T^2}.$$

→ Instantaneous rotational power : $\boxed{P = \vec{\tau} \cdot \vec{\omega}}$ $(P = \vec{F} \cdot \vec{v})$.

19. Rolling Motion

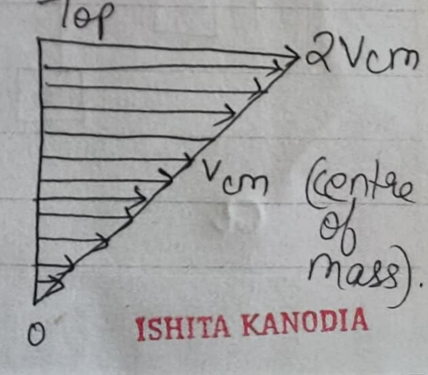
The velocity of the COM represents linear motion while angular velocity about the centroidal axis represents rotatory motion.

→ If the relative velocity of the point of contact of the rolling body with surface is zero $\Longrightarrow$ pure rolling. ☆☆.



$\boxed{V_{cm} = R\omega}$.

$\boxed{a_{cm} = R\alpha}$ or For pure rolling $\Longrightarrow$



Rotational KE $= \frac{1}{2} I \omega^2 = \frac{1}{2} m k^2 \frac{V^2}{R^2} = (E_{trans}) \frac{K^2}{R^2}$

Translational KE $= \frac{1}{2} m v^2$ Total KE $= E_{trans}\left(1 + \frac{K^2}{R^2}\right)$

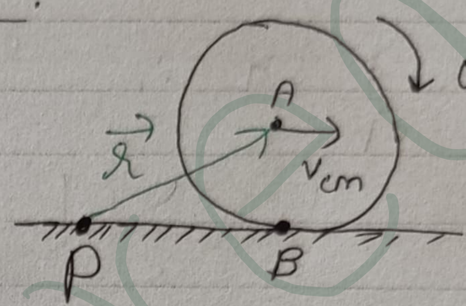
→ $E_{translational} : E_{rotation} : E_{rolling} = 1 : \frac{K^2}{R^2} : 1 + \frac{K^2}{R^2}$

21. Angular Momentum in Rolling

$\vec{L_p} = \vec{L_{cm}} + \vec{r} \times \vec{P_{cm}}$

$\vec{L_p} = \vec{L_{cm}} + \vec{r} \times m\vec{V_{cm}}$

$= I\vec{\omega_{cm}} + \vec{r} \times m\vec{V_{cm}}$

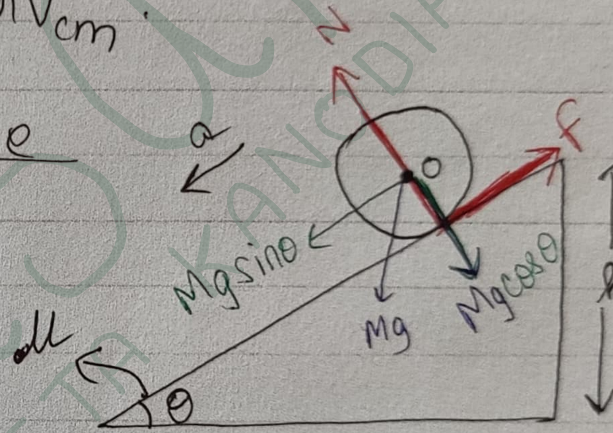


22. Rolling Motion On Incline

$Mg\sin\theta - f = Ma_{com}$

$\tau_o = fR = I\alpha$

$a_{cm} = R\alpha$ ⇒ pure rolling



For pure rolling – $\boxed{\mu \geq \frac{\tan\theta}{\left(1 + \frac{R^2}{k^2}\right)}}$

→ $V_{rolling}$ (at bottom of incline) $= \sqrt{\frac{2gh}{\left(1 + \frac{K^2}{R^2}\right)}}$ ($V_{sliding} > V_{rolling}$)

→ $a_{rolling} = \frac{g\sin\theta}{\left(1 + \frac{K^2}{R^2}\right)}$ ($a_{sliding} > a_{rolling}$)

→ $t_{rolling} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{K^2}{R^2}\right)}$ $t_{sliding} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}$

$t_{rolling} > t_{sliding}$.